MATH 579 Exam 2 Solutions

1. Let $a_0 = 3.1$, and let $a_{n+1} = \sqrt{a_n + 8}$ for $n > 0$. Prove that $3 < a_n < 3.5$ for all natural n.

   Induction on n. $n = 0$ holds since $3 < 3.1 < 3.5$. Suppose now that $3 < a_n < 3.5$. $f(x) = \sqrt{x+8}$ is a strictly increasing function, since $f'(x) = \frac{1}{2\sqrt{x+8}} > 0$. By the inductive hypothesis, $3 < a_n < 3.5$, and by the increasing property, $f(3) < f(a_n) < f(3.5)$. We have $f(3) = \sqrt{11} \approx 3.3166, f(3.5) = \sqrt{11.5} \approx 3.3912$, so in particular $3 < f(a_n) < 3.5$. We are done since $a_{n+1} = f(a_n)$.

2. For all positive integers n, prove that $\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$.

   Induction on n. $n = 1$ holds since $1 \cdot 2 \cdot 3 = \frac{1 \cdot 2 \cdot 3 \cdot 4}{4}$. We take as inductive hypothesis that $\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$, and add $(n+1)(n+2)(n+3)$ to both sides. The RHS factors as $(n+1)(n+2)(n+3)(\frac{n}{4}+1) = \frac{(n+1)(n+2)(n+3)(n+4)}{4}$, and we are done.

3. Let $a_0 = 2, a_n = 2a_{n-1} + 3^n$ (for $n \geq 1$). Prove that $a_n = 3^{n+1} - 2^n$.

   Induction on n. $n = 0$ holds since $2 = a_0 = 3^1 - 2^0$. We have as inductive hypothesis that $a_n = 3^{n+1} - 2^n$. Now $a_{n+1} = 2a_n + 3^{n+1} = 2(3^{n+1} - 2^n) + 3^{n+1} = 3 \cdot 3^{n+1} - 2 \cdot 2^n = 3^{n+2} - 2^{n+1}$, as desired.

4. Prove that a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

   Let $sum(n)$ denote the sum of the digits of n. We prove the stronger statement that $3|n - sum(n)$ for every natural n, by induction on the number of digits of n. If one digit, then $n = sum(n)$ so $3|0$ holds. Otherwise, write $n = 10x + a$. x is an integer with fewer digits than n, hence by the inductive hypothesis $3|x - sum(x)$. But also $3|9x$, so $3|x - sum(x) + 9x = 10x - sum(x) = (10x + a) - (sum(x) + a) = n - sum(n)$, as desired.

5. We have a rectangular chocolate bar, shaped as an $m \times n$ chessboard, from chocolate squares. We wish to break the bar into its mn constituent squares, iteratively: break the rectangle, take one of the resulting rectangles and break it, etc. Prove that it will take exactly $mn - 1$ breaks.

   We proceed using strong induction on the number of squares in the rectangle. If $m = n = 1$, no breaks are necessary, which is $1 \cdot 1 - 1$, as desired. Otherwise, we may assume without loss that we break our bar horizontally (rotating our bar if necessary), to divide it into rectangles of size $i \times n$ and $(m - i) \times n$. By the inductive hypothesis (once on each piece), these will take $in - 1$ and $(m - i)n - 1$ breaks, respectively, to reduce to squares. Hence all together there will be $1 + (in - 1) + ((m - i)n - 1) = mn - 1$ breaks.

   Alternate, clever, solution, avoiding induction entirely: Forget about squares, rectangles, m, n, and just count pieces. We begin with one piece of chocolate, and end with several. Each break, no matter how it's done, increases the number of pieces by one. Hence, the number of breaks must be (# of pieces at the end) $-$ (# of pieces at the beginning).

6. For all natural n, prove that $3^{(4 \times 10^n)}$ ends in $\ldots\underbrace{00\cdots0}_{n}1$. For example, for $n = 2$, 3^{400} ends in $\ldots 001$.

   Induction on n. $n = 0$ gives $3^4 = 81$, which ends in 1 as desired. We take as inductive hypothesis that $3^{(4 \times 10^n)} = 1 + s10^{n+1}$, for some integer s. Hence $3^{(4 \times 10^{n+1})} = \left(3^{(4 \times 10^n)}\right)^{10} = (1 + s10^{n+1})^{10} = 1^{10} + (10)1^9(s10^{n+1})^1 + (45)1^8(s10^{n+1})^2 + \cdots + (s10^{n+1})^{10}$, where we used Pascal's triangle or the binomial theorem or just multiplied the polynomial out. Note that every term after the first has a factor of 10^{n+2}, so we can write $3^{(4 \times 10^{n+1})} = 1 + 10^{n+2}t$, for some integer t. This completes the proof. Note: This problem was inspired by Exam 1, where several of you saw a pattern and made the (false) claim that $3^{(4 \times 5^n)}$ ends in $\ldots\underbrace{00\cdots0}_{n}1$.

Exam results: High score=104, Median score=76, Low score=56 (before any extra credit)